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Systems of equilibrium problems with applications to new variants of Ekeland's variational principle, fixed point theorems and parametric optimization problems

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Abstract In this paper, we first establish some existence theorems of systems of generalized vector equilibrium problems. From these results, we obtain new variants of Ekeland's variational principle in a Hausdorff t.v.s., a minimax theorem and minimization theorems. Some applications to the existence theorem of systems of semi-infinite problem, a variant of flower petal theorem and a generalization of Schauder's fixed point theorem are also given.

Keywords ℓ-function · Quasi-distance · Ekeland's variational principle · Flower petal theorem · Maximal element theorem · Equilibrium problem · Minimax theorem · Minimization theorem · Semi-infinite problem · Generalized Schauder's fixed point theorem

1 Introduction

The celebrated variational principle due to Ekeland [10] is an important tool in various fields of applied mathematical analysis and nonlinear analysis. Generalizations and variants of the Ekeland's variational principle were developed by several authors in different directions in the past; see [7, 8, 14–19, 21, 27–29, 32–34] and references therein. It is well-known that the original Ekeland's variational principle (in short EVP) [11–13, 19, 28, 34] is equivalent to the Caristi's fixed point theorem [5, 33], to the Takahashi's nonconvex minimization theorem [33], to the drop theorem [18, 30], and to the flower petal theorem [18, 30].

Let X be a nonempty subset of a topological space and $f: X \times X \to \mathbb{R}$ be a function with $f(x, x) \geq 0$ for all $x \in X$. Then, the scalar equilibrium problem is to find $\bar{x} \in X$ such that $f(\bar{x}, y) \geq 0$ for all $y \in X$. The equilibrium problem was extensively investigated and generalized to the vector equilibrium problems for single-valued or multivalued maps

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and contains optimization problems, variational inequalities problems, the Nash equilibrium problems, fixed point problems, complementary problems, bilevel problems and semi-infinite problems as special cases and applications; see [2, 4, 6, 13, 20, 22–25, 31] and references therein.

Till now, to our knowledge, almost all generalizations and applications on EVP are established on complete metric spaces or Banach spaces (if the convexity assumptions on sets or functions are needed). In this paper, we establish a variant of EVP on a topological vector space (in short t.v.s.) which is proved by an existence theorem of equilibrium problem. In our new variants of EVP, the functions we considered does not assume to be proper and bounded from below and we use quasi-distances instead of metrics (even other weakly (quasi-)metrics). Our variants of EVP are quite different from [7, 14–17, 34]. We give some applications to extend Schauder's fixed point theorem and obtain some common fixed point theorems. We also apply our variants of EVP to optimizational problem and give some equivalence relations between EVP, common fixed point theorem, maximal element theorem and minimization theorem.

The paper is divided into six sections. In Sect. 3, we first study a systems of generalized vector quasi-equilibrium problem, from which we establish some new variants of EVP in a Hausdorff t.v.s in Sect. 4. Our results and methods are quite different from [7, 8, 14–19, 21, 27–29, 32–34]. In Sect. 5, we establish some equivalent formulations of our theorems. Finally, in Sect. 6, we give some applications to study systems of semi-infinite problems, a variant of flower petal theorem, a generalization of Schauder's fixed point theorem and a minimax theorem. Our techniques and some results are quite original in the literatures.

2 Preliminaries

Throughout this paper, we denote the set of real numbers by $\mathbb{R}$. Let A and B be nonempty sets. A multivalued map $T: A \multimap B$ is a function from A to the power set 2^B of B. We denote $T(A) = \bigcup\{T(x): x \in A\}$ and let $T^-: B \multimap A$ be defined by the condition that $x \in T^-(y)$ if and only if $y \in T(x)$. Let X and Y be topological spaces. A multivalued map $T: X \multimap Y$ is said to be (i) upper semi-continuous (in short u.s.c.) at $x \in X$ if for every open set V in Y with $T(x) \subset V$, there exists an open neighborhood $U(x)$ of x such that $T(x') \subset V$ for all $x' \in U(x)$; (ii) lower semi-continuous (in short l.s.c.) at $x \in X$ if for every open set V in Y with $T(x) \bigcap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(x') \bigcap V \neq \emptyset$ for all $x' \in U(x)$; (iii) u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; (iv) closed if $GrT = \{(x, y) : x \in X, y \in T(x)\}$ is closed in $X \times Y$. (v) compact if there exists a compact set K such that $T(X) \subseteq K$.

Let Z be a real t.v.s. with zero vector θ, D a proper convex cone in Z and $A \subseteq Z$. A point $\bar{y} \in A$ is called a vectorial minimal point of A if for any $y \in A$, $y - \bar{y} \notin -D \setminus \{\theta\}$. The set of vectorial minimal point of A is denoted by $Min_D A$. The convex hull of A and the closure of A are denoted by coA and clA, respectively.

Definition 2.1 Let X be a nonempty convex subset of a vector space E, Y be a nonempty convex subset of a vector space V and Z be a real t.v.s. Let $F: X \times Y \multimap Z$ and $C: X \multimap Z$ be multivalued maps such that for each $x \in X$, $C(x)$ is a nonempty closed convex cone. For each fixed $x \in X$, $y \multimap F(x, y)$ is called $C(x)$-*quasiconvex* if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, we have either

$$F(x, y_1) \subseteq F(x, \lambda y_1 + (1 - \lambda) y_2) + C(x)$$

or

$$F(x, y_2) \subseteq F(x, \lambda y_1 + (1-\lambda) y_2) + C(x).$$

The following Lemmas and theorems are crucial in this paper.

Lemma 2.1 [1, 35] *Let X and Y be Hausdorff topological spaces, $T: X \multimap Y$ be a multivalued map. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there exists a subnet $\{x_{\phi(\lambda)}\}_{\lambda \in \Lambda}$ of $\{x_\alpha\}$ and a net $\{y_\lambda\}_{\lambda \in \Lambda}$ with $y_\lambda \to y$ such that $y_\lambda \in T(x_{\phi(\lambda)})$ for all $\lambda \in \Lambda$.*

Lemma 2.2 [26] *Let Z be a Hausdorff t.v.s. and C be a closed convex cone in Z. If A is a nonempty compact subset of Z, then* $\mathrm{Min}_C A \neq \emptyset$.

Lemma 2.3 [3] *Let X and Y be Hausdorff topological spaces, $T: X \multimap Y$ be a multivalued map.*

(i) If T is an u.s.c. multivalued map with closed values, then T is closed;
(ii) If Y is a compact space and T is closed, then T is u.s.c.;
(iii) If X is compact and T is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.

Theorem 2.1 [9] *Let I be any index set. Let $\{X_i\}_{i \in I}$ be a family of nonempty convex subsets, where each X_i is contained in a Hausdorff t.v.s. E_i. For each $i \in I$, let $S_i : X = \prod_{i \in I} X_i \multimap X_i$ be a multivalued map such that*

(i) for each $x = (x_i)_{i \in I} \in X$, $x_i \notin coS_i(x)$;
(ii) for each $y_i \in X_i$, $S_i^-(y_i)$ is open in X;
(iii) there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exists $j \in I$ such that $M_j \cap S_j(x) \neq \emptyset$.

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

3 Existence theorems of systems of generalized vector equilibrium problems

The following existence theorem of systems of equilibrium problems is one of the main results of this paper. It has many applications in variants of EVP in a Hausdorff t.v.s., semi-infinite problems, fixed point theorems, minimax theorems, and optimization problems.

Theorem 3.1 *Let I be any index set. For each $i \in I$, let X_i be a nonempty subset of a t.v.s. E_i, Y_i be a nonempty closed convex subset of a Hausdorff t.v.s. V_i, U_i and Z_i be real t.v.s. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $C_i : Y \multimap U_i$, $D_i : Y \multimap Z_i$, $F_i : X \times Y \multimap U_i$ and $G_i : Y \times Y_i \multimap Z_i$ be multivalued maps with nonempty values and $T_i : Y \multimap Y_i$ be a multivalued map with nonempty convex values. Let $u \in X$. For each $i \in I$, let $W_i = \{y \in Y : F_i(u, y) \subseteq C_i(y)\}$, $H_i = \{y_i \in Y_i : F_i(u, y) \subseteq C_i(y)$ for $y = (y_i)_{i \in I} \in Y\}$ and let $A_i : Y \multimap Y_i$ be defined by $A_i(y) = \{z_i \in Y_i : G_i(y, z_i) \nsubseteq D_i(y)\}$. For each $i \in I$, suppose that the following conditions are satisfied:*

(i) W_i is a nonempty closed subset of Y;

(ii) *for each* $y = (y_i)_{i\in I} \in Y$, $G_i(y, y_i) \subseteq D_i(y)$;
(iii) *for each* $y \in Y$, $T_i(y) \subseteq H_i$ *and* $A_i(y)$ *is convex;*
(iv) *for each* $z_i \in Y_i$, $T_i^-(z_i)$ *and* $A_i^-(z_i)$ *are open in* Y;
(v) *there exist a nonempty compact subset* K *of* Y *and a nonempty compact convex subset* M_i *of* Y_i *for each* $i \in I$ *such that for each* $y \in Y \setminus K$ *there exist* $j \in I$ *and* $z_j \in M_j \cap T_j(y)$ *such that* $G_j(y, z_j) \nsubseteq D_j(y)$.

Then, there exists $v \in Y$ such that for each $i \in I$, $F_i(u, v) \subseteq C_i(v)$ and $G_i(v, y_i) \subseteq D_i(v)$ for all $y_i \in T_i(v)$.

Proof For each $i \in I$, define a multivalued map $\varphi_i : Y \multimap Y_i$ by

$$\varphi_i(y) = \begin{cases} T_i(y) \cap A_i(y), & \text{if } y \in W_i \\ T_i(y), & \text{if } y \in Y \setminus W_i. \end{cases}$$

Then for each $i \in I$, $y_i \notin co\varphi_i(y)$ for all $y = (y_i)_{i\in I} \in Y$. Indeed, for each $i \in I$, if $y \in W_i$, then $\varphi_i(y) = T_i(y) \cap A_i(y) \subseteq A_i(y)$. By (iii), we have $co\varphi_i(y) \subseteq A_i(y)$. By (ii), we have $y_i \notin A_i(y)$ and hence $y_i \notin co\varphi_i(y)$. On the other hand, if $y \in Y \setminus W_i$, then $y_i \notin H_i$. By (iii) and the convexity of $T_i(y)$, we have $y_i \notin co\varphi_i(y)$. Hence for each $i \in I$, $y_i \notin co\varphi_i(y)$ for all $y = (y_i)_{i\in I} \in Y$. It is easy to see that for each $i \in I$ and $z_i \in Y_i$,

$$\varphi_i^-(z_i) = [T_i^-(z_i) \cap A_i^-(z_i)] \cup [(Y \setminus W_i) \cap T_i^-(z_i)].$$

Thus, from our hypothesis, $\varphi_i^-(z_i)$ is open in Y for each $(i, z_i) \in I \times Y_i$. By (v), there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \setminus K$ there exist $j \in I$, such that $M_j \cap \varphi_j(y) \neq \emptyset$. Applying Theorem 2.1, there exists $v \in Y$ such that $\varphi_i(v) = \emptyset$ for all $i \in I$. If $v \notin W_i$, then $\emptyset \neq T_i(v) = \varphi_i(v) = \emptyset$, which leads to a contradiction. Therefore, $v \in W_i$ and $y_i \notin A_i(v)$ for all $y_i \in T_i(v)$. The proof is completed. □

Corollary 3.1 *Let* I *be any index set. For each* $i \in I$, *let* X_i *be a nonempty subset of a t.v.s.* E_i, Y_i *be a nonempty closed convex subset of a Hausdorff t.v.s.* V_i, U_i *and* Z_i *be real t.v.s. Let* $X = \prod_{i\in I} X_i$ *and* $Y = \prod_{i\in I} Y_i$. *For each* $i \in I$, *let* $C_i : Y \multimap U_i$ *be a multivalued map with nonempty values,* $D_i : Y \multimap Z_i$ *be a closed multivalued map such that* $D_i(x)$ *is a nonempty convex cone for each* $x \in X$, $F_i : X \times Y \multimap U_i$ *and* $G_i : Y \times Y_i \multimap Z_i$ *be multivalued maps with nonempty values and* $T_i : Y \multimap Y_i$ *be a multivalued map with nonempty convex values. Let* $u \in X$. *For each* $i \in I$, *let* $W_i = \{y \in Y : F_i(u, y) \subseteq C_i(y)\}$, $H_i = \{y_i \in Y_i : F_i(u, y) \subseteq C_i(y), \text{for } y = (y_i)_{i\in I} \in Y\}$. *For each* $i \in I$, *suppose that*

(i) W_i *is a nonempty closed subset of* Y;
(ii) *for each* $y = (y_i)_{i\in I} \in Y$, $G_i(y, y_i) \subseteq D_i(y)$;
(iii) *for each* $y \in Y$, $T_i(y) \subseteq H_i$ *and for each* $z_i \in Y_i$, $T_i^-(z_i)$ *is open in* Y;
(iv) *for each* $y \in Y$, $G_i(y, \cdot)$ *is* $D_i(y)$*-quasiconvex and for each* $z_i \in Y_i$, $G_i(\cdot, z_i)$ *is l.s.c.;*
(v) *there exist a nonempty compact subset* K *of* Y *and a nonempty compact convex subset* M_i *of* Y_i *for each* $i \in I$ *such that for each* $y \in Y \setminus K$ *there exist* $j \in I$ *and* $z_j \in M_j \cap T_j(y)$ *such that* $G_j(y, z_j) \nsubseteq D_j(y)$.

Then there exists $v \in Y$ such that for each $i \in I$, $F_i(u, v) \subseteq C_i(v)$ and $G_i(v, y_i) \subseteq D_i(v)$ for all $y_i \in T_i(v)$.

Proof For each $i \in I$, let $A_i : Y \multimap Y_i$ be defined by

$$A_i(y) = \{z_i \in Y_i : G_i(y, z_i) \nsubseteq D_i(y)\}.$$

We first show that for each $(i, z_i) \in I \times Y_i$, $A_i^-(z_i)$ is open in Y. Let $y \in cl(Y \backslash A_i^-(z_i))$. Then there exists a net $\{y_\alpha\}_{\alpha \in \Lambda}$ in $Y \backslash A_i^-(z_i)$ such that $y_\alpha \to y$. Thus we have $G_i(y_\alpha, z_i) \subseteq D_i(y_\alpha)$. By the closedness of Y, $y \in Y$. Also, we obtain $G_i(y, z_i) \subseteq D_i(y)$. Indeed, for any $w \in G_i(y, z_i)$, since $G_i(\cdot, z_i)$ is l.s.c. at y and $y_\alpha \to y$, by Lemma 2.1, there exists a subnet $\{y_{\alpha_\lambda}\}_{\lambda \in \Lambda}$ of $\{y_\alpha\}$ and a net $\{w_\lambda\}_{\lambda \in \Lambda}$ with $w_\lambda \to w$ such that $w_\lambda \in G_i(y_{\alpha_\lambda}, z_i) \subseteq D_i(y_{\alpha_\lambda})$ for all $\lambda \in \Lambda$. Since D_i is closed, we have $w \in D_i(y)$. Thus, $G_i(y, z_i) \subseteq D_i(y)$. Therefore, $y \in Y \backslash A_i^-(z_i)$ and hence $A_i^-(z_i)$ is open in Y. Next, we claim that for each $(i, y) \in I \times Y$, $A_i(y)$ is convex. Let $a_i, b_i \in A_i(y)$. Then $G_i(y, a_i) \nsubseteq D_i(y)$ and $G_i(y, b_i) \nsubseteq D_i(y)$. By the convexity of Y_i, $e_i^{(\lambda)} := \lambda a_i + (1-\lambda) b_i \in Y_i$ for all $\lambda \in [0, 1]$. Suppose to the contrary that there exists $\lambda_0 \in (0, 1)$ such that $G_i(y, e_i^{(\lambda_0)}) \subseteq D_i(y)$. By the $D_i(y)$-quasiconvexity of $G_i(y, \cdot)$, either

$$G_i(y, a_i) \subseteq G_i(y, e_i^{(\lambda_0)}) + D_i(y) \subseteq D_i(y)$$

or

$$G_i(y, b_i) \subseteq G_i(y, e_i^{(\lambda_0)}) + D_i(y) \subseteq D_i(y).$$

This leads to a contradiction. Hence for each $(i, y) \in I \times Y$, $A_i(y)$ is convex. Therefore, all the conditions of Theorem 3.1 are satisfied and the conclusion follows from Theorem 3.1. □

Using the same argument in the proof of Corollary 3.1, we have the following result.

Lemma 3.1 *Let X be a t.v.s. and U be a real t.v.s. Let $F: X \times X \multimap U$ be a multivalued map with nonempty values and $C: X \multimap U$ be a closed multivalued map with nonempty values. Let $u \in X$. If there exists $w = w(u) \in X$ such that $F(u, w) \subseteq C(w)$ and $F(u, \cdot)$ is l.s.c., then $W = \{x \in X : F(u, x) \subseteq C(x)\}$ is a nonempty closed subset of X.*

Note that $\mathbb{R}^+ := [0, \infty)$ and $\mathbb{R}^- := (-\infty, 0]$ are closed convex cones in $(-\infty, \infty]$. From Lemma 3.1 and Corollary 3.1, we have the following result.

Corollary 3.2 *Let I be any index set. For each $i \in I$, let X_i be a nonempty subset of a t.v.s. E_i, Y_i be a nonempty closed convex subset of a Hausdorff t.v.s. V_i. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $F_i : X \times Y \multimap (-\infty, \infty]$ and $G_i : Y \times Y_i \multimap (-\infty, \infty]$ be multivalued maps with nonempty values and $T_i : Y \multimap Y_i$ be a multivalued map with nonempty convex values, and let $H_i = \{y_i \in Y_i : F_i(u, y) \subseteq \mathbb{R}^-$, for $y = (y_i)_{i \in I} \in Y\}$. Let $u \in X$. For each $i \in I$, suppose that there exists $w = w(i, u) \in Y$ such that $F_i(u, w) \subseteq \mathbb{R}^-$. For each $i \in I$, suppose that*

- *(i) $F(u, \cdot)$ is l.s.c.;*
- *(ii) for each $y = (y_i)_{i \in I} \in Y$, $G_i(y, y_i) \subseteq \mathbb{R}^+$;*
- *(iii) for each $y \in Y$, $T_i(y) \subseteq H_i$ and for each $z_i \in Y_i$, $T_i^-(z_i)$ is open in Y;*
- *(iv) for each $y \in Y$, $G_i(y, \cdot)$ is $\mathbb{R}^+$-quasiconvex and for each $z_i \in Y_i$, $G_i(\cdot, z_i)$ is l.s.c.;*
- *(v) there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $x \in Y \backslash K$ there exist $j \in I$ and $z_j \in M_j \cap T_j(y)$ such that $G_j(y, z_j) \nsubseteq \mathbb{R}^+$.*

Then there exists $v \in Y$ such that $\sup F_i(u, v) \leq 0$ and $\inf G_i(v, T_i(v)) \geq 0$ for all $i \in I$.

The following result is a powerful tool to establish variants of EVP in a Hausdorff t.v.s in Sect. 4.

Theorem 3.2 *Let X be a nonempty subsets of a t.v.s. E and Y be a Hausdorff t.v.s. Let $f: X \times Y \to (-\infty, \infty]$ and $g: Y \times Y \to (-\infty, \infty]$ be functions. Let $u \in X$. Suppose that*

(i) *$W = \{y \in Y\colon f(u, y) \le 0\}$ is a nonempty closed convex subset of Y;*
(ii) *for each $y \in Y$, $g(y, y) \ge 0$;*
(iii) *for each $x \in Y$, $g(x, \cdot)$ is quasiconvex and for each $y \in Y$, $g(\cdot, y)$ is u.s.c.;*
(iv) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M of Y such that for each $y \in Y \setminus K$ there exists $z \in M$ such that $f(u, z) \le 0$ and $g(y, z) < 0$.*

Then there exists $v \in Y$ such that $f(u, v) \le 0$ and $g(v, y) \ge 0$ for all $y \in W$.

Proof Let $U = Z = (-\infty, \infty]$ and $C, D: Y \multimap (-\infty, \infty]$ be defined by $C(y) = \mathbb{R}^-$ and $D(y) = \mathbb{R}^+$ for all $y \in Y$. Let $T: Y \multimap Y$ be defined by

$$T(y) = W \ \text{ for all } y \in Y$$
$$\Longleftrightarrow \quad T^-(z) = \begin{cases} Y, & \text{if } z \in W \\ \emptyset, & \text{if } z \in Y \setminus W. \end{cases}$$

and $A: Y \multimap Y$ defined by

$$A(x) = \{y \in Y: \ g(x, y) < 0\}.$$

Then $T^-(z)$ is open in Y for all $z \in Y$. By (i), $T(y) = W$ is a nonempty closed convex subset of Y for all $y \in Y$. By (ii), $g(y, y) \in D(y)$ for each $y \in Y$. By (iii), for each $y \in Y$, $A^-(y) = \{x \in Y: \ g(x, y) < 0\}$ is open in Y and for each $x \in Y$, $A(x)$ is convex. Therefore, all the conditions of Theorem 3.1 are satisfied and the conclusion follows from Theorem 3.1. □

4 Variants of EVP in Hausdorff t.v.s.

In this section, we first introduce the concepts of ℓ-function and quasi-distance.

Definition 4.1 Let X be a t.v.s. A function $p: X \times X \to (-\infty, \infty]$ is called

(a) a *ℓ-function* if the following are satisfied:
  ($L1$) $p(x, x) \ge 0$ for all $x \in X$;
  ($L2$) for any $x \in X$, $p(x, \cdot)$ is convex;
  ($L3$) for any $y \in X$, $p(\cdot, y)$ is u.s.c.
(b) a *quasi-distance* on X if the following are satisfied:
  ($QD1$) $p(x, x) \ge 0$ for all $x \in X$;
  ($QD2$) $p(x, z) \le p(x, y) + p(y, z)$ for any $x, y, z \in X$;
  ($QD3$) for any $x \in X$, $p(x, \cdot)$ is convex and l.s.c.;
  ($QD4$) for any $y \in X$, $p(\cdot, y)$ is u.s.c.

Obviously, a quasi-distance is a ℓ-function, but the converse is not true. It is easy to see that if p_1 and p_2 are quasi-distances (resp. ℓ-functions) and $\alpha \ge 0$, then αp_1 and $p_1 + p_2$ are quasi-distances (resp. ℓ-functions).

Example A

(a) Let X be a Hausdorff t.v.s. and $f: X \to (-\infty, \infty]$ be a l.s.c. and convex function. Then, the function $p: X \times X \to (-\infty, \infty]$ defined by $p(x, y) = f(y) - f(x)$ is a quasi-distance on X.

(b) Let H be an inner product space equipped with a inner product $\langle\cdot,\cdot\rangle$ and $T: H \to H$ be a continuous map. Then, the function $p: X \times X \to [0,\infty)$ defined by $p(x,y) = \langle y-x, Tx\rangle$ is a ℓ-function.

(c) Let $(X, \|\cdot\|)$ be a normed vector space.

(1) The function $p: X \times X \to [0,\infty)$ defined by

$$p(x,y) = a\,\|x-y\| + f(x) + g(y)$$

is a quasi-distance on X, where $a \geq 0$, $f: X \to [0,\infty)$ is a u.s.c. function and $g: X \to [0,\infty)$ is a convex and l.s.c. function. In particular, any constant function on $X \times X$, the function $p: X \times X \to [0,\infty)$ defined by $p(x,y) = \|x\| + \|y\|$ and the metric $d(x,y) = \|x-y\|$ are quasi-distances on X;

(2) The function $p: X \times X \to [0,\infty)$ defined by

$$p(x,y) = \max\{\|Tx-y\|, \|Tx-Ty\|\}$$

is a quasi-distance on X, where $T: X \to X$ is an affine continuous map. Indeed, $(QD1)$ holds clearly. For any $x, y, z \in X$, since

$$\begin{aligned} p(x,z) &= \max\{\|Tx-z\|, \|Tx-Tz\|\} \\ &\leq \max\{\|Tx-y\|, \|Tx-Ty\|\} + \max\{\|Ty-z\|, \|Ty-Tz\|\} \\ &= p(x,y) + p(y,z), \end{aligned}$$

this shows that $(QD2)$ holds. Since

$$\begin{aligned} p(x,y) &= \max\{\|Tx-y\|, \|Tx-Ty\|\} \\ &= \frac{|\|Tx-y\| - \|Tx-Ty\|| + \|Tx-y\| + \|Tx-Ty\|}{2} \end{aligned}$$

and by the continuity of T and $\|\cdot\|$, $p: X \times X \to [0,\infty)$ is a continuous function. Hence $(QD4)$ holds. For any $x \in X$, we claim that $p(x,\cdot)$ is convex. Let y_1, $y_2 \in X$ and $\lambda \in [0,1]$. Since T is affine, we have

$$\begin{aligned} &p(x, \lambda y_1 + (1-\lambda)y_2) \\ &\quad \leq \max\{\lambda\,\|Tx-y_1\| + (1-\lambda)\,\|Tx-y_2\|, \\ &\qquad\qquad \lambda\,\|Tx-Ty_1\| + (1-\lambda)\,\|Tx-Ty_2\|\} \\ &\quad \leq \lambda p(x,y_1) + (1-\lambda)p(x,y_2). \end{aligned}$$

So $(QD3)$ holds. Therefore p is a quasi-distance on X.

The following result is a variant of EVP for quasi-distances in a Hausdorff t.v.s. It is quite different from EVP in metric spaces or quasi-metric spaces.

Theorem 4.1 *Let X be a Hausdorff t.v.s. Let $f: X \to (-\infty,\infty]$ be a l.s.c. and convex function and $p: X \times X \to (-\infty,\infty]$ be a quasi-distance. Let $u \in X$ with $p(u,u) = 0$ and $\varepsilon > 0$. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $\varepsilon p(u,z) \leq f(u) - f(z)$ and $\varepsilon p(y,z) < f(y) - f(z)$. Then there exists $v \in X$ such that*

(i) $\varepsilon p(u,v) \leq f(u) - f(v)$;
(ii) $\varepsilon p(v,x) \geq f(v) - f(x)$ for all $x \in X$.

Proof Since p is a quasi-distance, εp is also a quasi-distance. Define h, $g: X \times X \to (-\infty, \infty]$ by

$$h(x, y) = g(x, y) = \varepsilon p(x, y) - f(x) + f(y).$$

By the lower semi-continuity and the convexity of f and $p(u, \cdot)$,

$$W := \{x \in X : h(u, x) \le 0\} = \{x \in X : \varepsilon p(u, x) \le f(u) - f(x)\}$$

is a nonempty closed convex subsets of X. Clearly, $g(x, x) \ge 0$ for each $x \in X$. By the upper semi-continuity of $-f$ and $p(\cdot, y)$, the function $x \to g(x, y)$ is u.s.c. for all $y \in X$. By the convexity of f and $p(x, \cdot)$, the function $y \to g(x, y)$ is convex for all $x \in X$. By our hypothesis, there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $h(u, z) \le 0$ and $g(y, z) < 0$. Therefore, all the conditions of Theorem 3.2 are satisfied. Thus there exists $v \in X$ such that

(i) $\varepsilon p(u, v) \le f(u) - f(v)$;
(ii) $\varepsilon p(v, x) \ge f(v) - f(x)$ for all $x \in W$.

For any $x \in X \setminus W$, since

$$\begin{aligned}\varepsilon[p(u, v) + p(v, x)] &\ge \varepsilon p(u, x)\\ &> f(u) - f(x)\\ &\ge \varepsilon p(u, v) + f(v) - f(x),\end{aligned}$$

it follows that $\varepsilon p(v, x) > f(v) - f(x)$ for all $x \in X \setminus W$. Hence $\varepsilon p(v, x) \ge f(v) - f(x)$ for all $x \in X$. The proof is completed. □

Remark 4.1

(a) Using Example A and Theorem 4.1, we can obtain several different variants of EVP in different spaces.
(b) Our variant of EVP (Theorem 4.1) is quite different from [7, 8, 14–19, 21, 27–29, 32–34]. Theorem 4.1 is comparable to the original EVP in the following aspects:
   (1) In Theorem 4.1, the functions we considered are defined on a Hausdorff t.v.s., but in the original EVP, the functions are defined on a complete metric space;
   (2) In the original EVP, the function f is assumed to be proper, l.s.c. and bounded from below, but in Theorem 4.1, f is not assumed to be proper and bounded from below. We only assume that $f: X \to (-\infty, \infty]$ is a l.s.c. and convex function;
   (3) In Theorem 4.1, we use quasi-distances instead of metrics (even other weakly (quasi-)metrics).

The following results are variants of EVP for ℓ-functions in a Hausdorff t.v.s.

Theorem 4.2 *Let X be a Hausdorff t.v.s. Let $f: X \to (-\infty, \infty]$ be a l.s.c. and convex function, $p: X \times X \to (-\infty, \infty]$ be a ℓ-function and $\varepsilon > 0$. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $\varepsilon p(y, z) < f(y) - f(z)$. Then, there exists $v \in X$ such that $\varepsilon p(v, x) \ge f(v) - f(x)$ for all $x \in X$.*

Proof Define $k: X \times X \to (-\infty, \infty]$ by $k(x, y) = 0$ for all $(x, y) \in X \times X$. Then for any $x \in X$, $W := \{y \in X : k(x, y) \le 0\} = X$. Let $g: X \times X \to (-\infty, \infty]$ be the same as in

Theorem 4.1. Using the same argument in the proof of Theorem 4.1, one can verify that all the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, there exists $v \in X$ such that $\varepsilon p(v, x) \geq f(v) - f(x)$ for all $x \in X$. □

Remark 4.2 Although a quasi-distance is a ℓ-function, Theorem 4.1 is not a special case of Theorem 4.2 because the conclusion (i) in Theorem 4.1 can't be deduced by Theorem 4.2.

Theorem 4.3 *Let X be a Hausdorff t.v.s. Let $f: X \to (-\infty, \infty]$ be a l.s.c. and convex function, $p : X \times X \to (-\infty, \infty]$ be a ℓ-function and $\varepsilon > 0$. Let $u \in X$. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $f(z) \leq f(u)$ and $\varepsilon p(y, z) < f(y) - f(z)$. Then, there exists $v = v(u) \in X$ such that*

(i) $f(v) \leq f(u)$;
(ii) $\varepsilon p(v, x) \geq f(v) - f(x)$ for all $x \in \{z \in X : f(z) \leq f(u)\}$.

Proof Let $k' : X \times X \to (-\infty, \infty]$ be defined by $k'(x, y) = f(y) - f(x)$ and let g: $X \times X \to (-\infty, \infty]$ be the same as in Theorem 4.1, then, by the lower semi-continuity and the convexity of f,

$$W := \{z \in X : k'(u, z) \leq 0\} = \{z \in X : f(z) \leq f(u)\}$$

is a nonempty closed convex subsets of X. Hence, the conclusion follows from Theorem 3.2. □

The following theorem is a special case of Theorem 5.1 in [24].

Theorem 4.4 *Let X be a Hausdorff t.v.s. and p: $X \times X \to (-\infty, \infty]$ be a ℓ-function. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $p(y, z) < 0$. Then, there exists $v \in X$ such that $p(v, x) \geq 0$ for all $x \in X$.*

Proof Take $c \in \mathbb{R}$ and let $\varepsilon = 1$. Define f: $X \to (-\infty, \infty]$ by $f(x) = c$ for all $x \in X$. Then f is a l.s.c. and convex function. By Theorem 4.2, there exists $v \in X$ such that $p(v, x) \geq f(v) - f(x) = 0$ for all $x \in X$. □

Remark 4.3 It is easy to see that Theorems 4.2 and 4.4 are equivalent.

5 Equivalent formulations of EVP

Definition 5.1 Let X be a t.v.s., f: $X \to (-\infty, \infty]$ and p: $X \times X \to (-\infty, \infty]$ are functions. We call that X satisfies (p, f)-*condition* if there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $p(y, z) < f(y) - f(z)$.

Below, unless otherwise specified, we shall assume that X is a Hausdorff t.v.s., f: $X \to (-\infty, \infty]$ is a l.s.c. and convex function, p: $X \times X \to (-\infty, \infty]$ is a ℓ-function and X satisfies (p, f)-condition. In this section, we establish some existence theorems in Hausdorff t.v.s. equipped with (p, f)-condition and prove that these theorems are equivalent to Theorem 4.2.

Theorem 5.1 (Common fixed point theorem for a family of multivalued maps) *Let I be an index set. For each $i \in I$, let $T_i : X \multimap X$ be a multivalued map with nonempty values such that for each $(i, x) \in I \times X$ with $x \notin T_i(x)$, there exists $y = y(x, i) \in X$ with $y \neq x$ such that $p(x, y) < f(x) - f(y)$. Then there exists $x_0 \in X$ such that $x_0 \in \bigcap_{i \in I} T_i(x_0)$. That is, $\{T_i\}_{i \in I}$ has a common fixed point in X.*

Theorem 5.2 (Common fixed point theorem for a family of single-valued maps) *Let I be an index set. For each $i \in I$, suppose that $T_i : X \to X$ is a map satisfying*

$$p(x, T_i x) < f(x) - f(T_i x)$$

for all $x \neq T_i(x)$. Then there exists $x_0 \in X$ such that $T_i(x_0) = x_0$ for all $i \in I$.

Theorem 5.3 (Maximal element theorem for a family of multivalued maps) *Let I be an index set. For each $i \in I$, let $T_i : X \multimap X$ be a multivalued map. Suppose that for each $(x, i) \in X \times I$ with $T_i(x) \neq \emptyset$, there exists $y = y(x, i) \in X$ with $y \neq x$ such that $p(x, y) < f(x) - f(y)$. Then there exists $x_0 \in X$ such that $T_i(x_0) = \emptyset$ for all $i \in I$.*

Theorem 5.4 *Theorem 4.2, Theorems 5.1, 5.2 and 5.3 are equivalent.*

Proof

(1). " Theorem 4.2$\Longleftrightarrow$ Theorem 5.1".
($\Rightarrow$) Applying Theorem 4.2, there exists $v \in X$ such that $p(v, x) \geq f(v) - f(x)$ for all $x \in X$. We want to show that $v \in \bigcap_{i \in I} T_i(v)$. If $v \notin T_{i_0} v$ for some $i_0 \in I$, then, by hypothesis, there exists $w(v, i_0) \in X$ with $w(v, i_0) \neq v$ such that $p(v, w(v, i_0)) < f(v) - f(w(v, i_0)) \leq p(v, w(v, i_0))$, which leads to a contradiction. Hence $v \in T_i(v)$ for all $i \in I$ and v is a common fixed point of $\{T_i\}_{i \in I}$.
($\Leftarrow$) Suppose that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) < f(x) - f(y)$. Then for each $x \in X$, we can define a multivalued map $T : X \multimap X \setminus \{\emptyset\}$ by

$$T(x) = \{y \in X : \ p(x, y) < f(x) - f(y)\}.$$

Clearly, $x \notin T(x)$ for all $x \in X$. By Theorem 5.1, T have a fixed point v in X, i.e., $v \in T(v)$. Hence we obtain a contradiction and Theorem 4.2 holds.

(2). " Theorem 5.1$\Longleftrightarrow$Theorem 5.2".
($\Rightarrow$) Under the assumption of Theorem 5.2, for each $i \in I$, let $G_i : X \multimap X$ be defined by $G_i(x) = \{T_i(x)\}$. Then for each $(i, x) \in I \times X$ with $x \notin G_i(x)$, we have $x \neq T_i(x)$. By hypothesis, $p(x, T_i x) < f(x) - f(T_i x)$ and hence, by Theorem 5.1, there exists $v \in X$ such that $v \in \bigcap_{i \in I} G_i(v)$ or $T_i v = v$ for all $i \in I$. This shows that Theorem 5.1 implies Theorem 5.2.
($\Leftarrow$) Under the assumption of Theorem 5.1, for each $(i, x) \in I \times X$ with $x \notin T_i(x)$, there exists $y(x, i) \in X$ with $y(x, i) \neq x$ such that $p(x, y(x, i)) < f(x) - f(y(x, i))$. Then one can define $g_i : X \to X$ by

$$g_i(x) = \begin{cases} x, & \text{if } x \in T_i(x) \\ y(x, i), & \text{if } x \notin T_i(x). \end{cases}$$

Hence g_i is a selfmap of X into X satisfying $p(x, g_i(x)) < f(x) - f(g_i(x))$ for all $x \neq g_i(x)$. By Theorem 5.2, there exists $v \in X$ such that $v = g_i(v) \in T_i(v)$ for all $i \in I$. This shows that Theorem 5.2 implies Theorem 5.1.

(3). " Theorem 4.2$\Longleftrightarrow$ Theorem 5.3".
($\Rightarrow$) Applying Theorem 4.2, there exists $v \in X$ such that $p(v, x) \geq f(v) - f(x)$ for all $x \in X$. We claim that $T_i(v) = \emptyset$ for all $i \in I$. Suppose to the contrary that there exists $i_0 \in I$ such that $T_{i_0}(v) \neq \emptyset$. By hypothesis, there exists $w = w(v, i_0) \in X$ with $w \neq v$ such that $p(v, w) < f(v) - f(w)$. Then, it leads to a contradiction. Therefore $T_i(v) = \emptyset$ for all $i \in I$.

($\Leftarrow$) Suppose that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) < f(x) - f(y)$. For each $x \in X$, define a multivalued map $T : X \multimap X \setminus \{\emptyset\}$ by

$$T(x) = \{y \in X : p(x, y) < f(x) - f(y)\}.$$

Then $T(x) \neq \emptyset$ for all $x \in X$. But applying Theorem 5.3, there exists $x_0 \in X$ such that $T(x_0) = \emptyset$. This is a contradiction. □

The following minimization theorem is immediate from Theorem 4.2.

Theorem 5.5 (Minimization theorem) *Suppose that for any $x \in X$ with $f(x) > \inf_{z \in X} f(z)$ there exists $y \in X$ with $y \neq x$ such that $p(x, y) < f(x) - f(y)$ holds. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{z \in X} f(z)$.*

Proof Applying Theorem 4.2, there exists $v \in X$ such that $p(v, x) \geq f(v) - f(x)$ for all $x \in X$. We claim that $f(v) = \inf_{z \in X} f(z)$. Suppose to the contrary that $f(v) > \inf_{x \in X} f(x)$. By our assumption, there exists $y = y(v) \in X$ with $y \neq v$ such that $p(v, y) < f(v) - f(y) \leq p(v, y)$, which leads to a contradiction. Therefore $f(v) = \inf_{z \in X} f(z)$. □

Remark 5.1

(a) Review the proof of Theorem 5.4, one can obtain a $v \in X$ such that
   (1) $p(v, x) \geq f(v) - f(x)$ for all $x \in X$;
   (2) $v \in T_i(v)$ (T_i is defined as in Theorem 5.1) or $T_i(v) = v$ (T_i is defined as in Theorem 5.2) for all $i \in I$;
   (3) $T_i(v) = \emptyset$ (T_i is defined as in Theorem 5.3) for all $i \in I$.

(b) Theorem 4.2 and Theorem 5.5 are equivalent if one further adds the condition "$p(x, y) \geq 0$ for all $x, y \in X$". Indeed, it suffices to show that Theorem 5.5 implies Theorem 4.2. Suppose that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) < f(x) - f(y)$. Then, by Theorem 5.5, there exists $v \in X$ such that $f(v) = \inf_{x \in X} f(x)$. By our hypothesis, there exists $w \in X$ with $w \neq v$ such that $p(v, w) < f(v) - f(w) \leq 0$, which leads to a contradiction.

6 Some applications

Applying Theorem 3.1, we have the following existence theorem of systems of semi-infinite problem.

Theorem 6.1 *Let I be any index set. For each $i \in I$, let X_i be a nonempty subset of a t.v.s. E_i, Y_i be a nonempty closed convex subset of a Hausdorff t.v.s. V_i. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $f_i : X \times Y \to (-\infty, \infty]$ and $g_i : Y \times Y_i \to (-\infty, \infty]$ be functions, $T_i : Y \multimap Y_i$ be a multivalued map with nonempty convex values, and let $H_i = \{y_i \in Y_i : f_i(u, y) \leq 0, \text{for } y = (y_i)_{i \in I} \in Y\}$. Let $u \in X$. For each $i \in I$, suppose that there exists $w = w(i, u) \in Y$ such that $f_i(u, w) \leq 0$. For each $i \in I$, suppose that*

(i) $f_i(u, \cdot)$ is l.s.c.;
(ii) for each $y = (y_i)_{i \in I} \in Y$, $g_i(y, y_i) \geq 0$;
(iii) for each $y \in Y$, $T_i(y) \subseteq H_i$ and for each $z_i \in Y_i$, $T_i^-(z_i)$ is open in Y;
(iv) for each $y \in Y$, $g_i(y, \cdot)$ is quasiconvex and g_i: $Y \times Y_i \to (-\infty, \infty]$ is u.s.c.;
(v) there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \setminus K$ there exist $j \in I$ and $z_j \in M_j \cap T_j(y)$ such that $g_j(y, z_j) < 0$.

If h: $Y \multimap Z_0$ is an u.s.c. multivalued map with nonempty compact values, where Z_0 is a real t.v.s. ordered by a proper closed convex cone C in Z_0, then there exists an optimal solution to the following problem (P):

$$\begin{aligned}&\mathrm{Min}_C h(y)\\ &\text{object to } y \in Y,\ f_i(u, y) \le 0 \text{ and } g_i(y, z_i) \ge 0\\ &\quad \text{for all } z_i \in T_i(y) \text{ and for all } i \in I.\end{aligned} \tag{1}$$

Proof For each $i \in I$, let

$$N_i = \{y \in Y : f_i(u, y) \le 0 \text{ and } g_i(y, z_i) \ge 0 \text{ for all } z_i \in T_i(y)\}.$$

Then N_i is closed in Y for all $i \in I$. Indeed, for each $i \in I$, let $y_i \in clN_i$. Then there exists a net $\{y_i^\alpha\}_{\alpha\in\Lambda}$ in N_i such that $y_i^\alpha \to y_i$. Hence $f_i(u, y_i^\alpha) \le 0$ and $g_i(y_i^\alpha, z_i) \ge 0$ for all $z_i \in T_i(y_i^\alpha)$. Let $a_i \in T_i(y)$. Since $T_i^-(z_i)$ is open in Y for each $z_i \in Y_i$, T_i is l.s.c. Hence there exists a net $\{a_i^\alpha\}_{\alpha\in\Lambda}$ with $a_i^\alpha \to a_i$ such that $a_i^\alpha \in T_i(y_i^\alpha)$. So $f_i(u, y_i^\alpha) \le 0$ and $g_i(y_i^\alpha, a_i^\alpha) \ge 0$. By (i), we have $f_i(u, y_i) \le 0$. By (iv), we have $g_i(y_i, a_i) \ge 0$. Hence $y_i \in N_i$ and N_i is a closed set in Y. Let $N = \cap_{i\in I} N_i$. Then N is closed in Y. Applying Theorem 3.1, $N \ne \emptyset$. By (v), it is easy to see that $N \subseteq K$, where K is a nonempty compact subset of Y in condition (v). Hence N is a nonempty compact subset of Y. Since $h: Y \multimap Z_0$ is an u.s.c. multivalued map with nonempty compact values, it follows from Lemma 2.3 that $h(N)$ is compact. Then by Lemma 2.2 that $\mathrm{Min}_C h(N) \ne \emptyset$. That is there exists a solution to the problem (P). The proof is completed. □

Theorem 6.2 *In Theorem 6.1, if we assume that $h: Y \to (-\infty, \infty]$ is a l.s.c. function, then there exists an optimal solution to the problem (P) as in Theorem 6.1.*

Proof Let N be the same as in the proof of Theorem 6.1. By the lower semi-continuity of h and the compactness of N, there exists $v \in N$ such that $h(v) = \min h(N)$. The proof is completed. □

Definition 6.1 Let X be a Hausdorff t.v.s. and $p: X \times X \to (-\infty, \infty]$ be a quasi-distance. The (p, ε)-$flower\ petal$ associated with $\varepsilon \in (0, \infty)$ and $a, b \in X$ (in short, $GP_\varepsilon[a, b]$) is the closed set

$$GP_\varepsilon[a, b] = \{x \in X : \varepsilon p(a, x) \le p(b, a) - p(b, x)\}.$$

We denote

$$P(\varepsilon, a, b) = \{x \in X : \varepsilon p(a, x) < p(b, a) - p(b, x)\}.$$

Obviously, $GP_\varepsilon[a, b]$ and $P(\varepsilon, a, b)$ are convex and if the quasi-distance p with $p(a, a) = 0$, then $GP_\varepsilon[a, b]$ is nonempty.

Theorem 6.3 (New variant of flower petal theorem) *Let X be a Hausdorff t.v.s. Let $a, b \in X$ and $\varepsilon > 0$. Let $p: X \times X \to (-\infty, \infty]$ be a quasi-distance with $p(a, a) = 0$. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $\varepsilon p(u, z) \le p(b, u) - p(b, z)$ and $\varepsilon p(y, z) < p(b, y) - p(b, z)$. Then there exists $v \in GP_\varepsilon[a, b]$ such that $P(\varepsilon, v, b) = \emptyset$.*

Proof Define f: $X \to (-\infty, \infty]$ by $f(x) = p(b, x)$. Then f is a l.s.c. and convex function. By Theorem 4.1, there exists $v \in X$ such that

(1) $\varepsilon p(a, v) \le f(a) - f(v)$;

(2) $\varepsilon p(v, x) \geq f(v) - f(x)$ for all $x \in X$.

By (1), we have $v \in GP_\varepsilon[a, b]$. By (2), we obtain $\varepsilon p(v, x) \geq p(b, v) - p(b, x)$ for all $x \in X$. Hence $x \notin P(\varepsilon, v, b)$ for all $x \in X$ or $P(\varepsilon, v, b) = \emptyset$. The proof is completed. □

The following minimax theorem is established by Theorem 4.2.

Theorem 6.4 (Minimax theorem) *Let X be a Hausdorff t.v.s. and $p\colon X \times X \to (-\infty, \infty]$ be a ℓ-function. Let $F : X \times X \to (-\infty, \infty]$ be a function satisfying for each $y \in X$, the function $x \to F(x, y)$ is l.s.c. and convex. Suppose that*

(i) for each $x \in X$ with $\{u \in X\colon F(x, u) > \inf_{a\in X} F(a, u)\} \neq \emptyset$, there exists $y = y(x) \in X$ with $y \neq x$ such that

$$p(x, y) < F(x, w) - F(y, w) \quad \text{for all } w \in X;$$

(ii) there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $p(y, z) < F(y, w) - F(z, w)$ for all $w \in X$.

Then $\inf_{x\in X} \sup_{y\in X} F(x, y) = \sup_{y\in X} \inf_{x\in X} F(x, y)$.

Proof Applying Theorem 4.2, for each $z \in X$, there exists $v(z) \in X$ such that $p(v(z), x) \geq F(v(z), z) - F(x, z)$ for all $x \in X$. Let $\gamma = \sup_{y\in X} \inf_{x\in X} F(x, y)$. Then $\inf_{x\in X} F(x, y) \leq \gamma$ for all $y \in X$. We first show that $\bigcap_{y\in X}\{x \in X\colon F(x, y) \leq \gamma\} \neq \emptyset$. Suppose to the contrary that $\bigcap_{y\in X}\{x \in X\colon F(x, y) \leq \gamma\} = \emptyset$. Then $v(z) \notin \bigcap_{y\in X}\{x \in X\colon F(x, y) \leq \gamma\}$ for all $z \in X$. Hence there exists $w_0 = w(v(z)) \in X$ such that $F(v(z), w_0) > \inf_{a\in X} F(a, w_0)$ or $w_0 \in \{u \in X\colon F(v(z), u) > \inf_{a\in X} F(a, u)\}$. So, $\{u \in X\colon F(v(z), u) > \inf_{a\in X} F(a, u)\} \neq \emptyset$ for all $z \in X$. Hence for each $z \in X$, there exists $y = y(v(z)) \in X$ with $y \neq v(z)$ such that $p(v(z), y) < F(v(z), w) - F(y, w)$ for all $w \in X$. This leads to a contradiction. Hence $\bigcap_{y\in X}\{x \in X\colon F(x, y) \leq \gamma\} \neq \emptyset$. Let $c \in \bigcap_{y\in X}\{x \in X\colon F(x, y) \leq \gamma\}$. Then, $\sup_{y\in X} F(c, y) \leq \gamma$. It follows that

$$\inf_{x\in X} \sup_{y\in X} F(x, y) \leq \sup_{y\in X} F(c, y) \leq \sup_{y\in X} \inf_{x\in X} F(x, y).$$

Since $\sup_{y\in X} \inf_{x\in X} F(x, y) \leq \inf_{x\in X} \sup_{y\in X} F(x, y)$ is always true, we show that $\inf_{x\in X} \sup_{y\in X} F(x, y) = \sup_{y\in X} \inf_{x\in X} F(x, y)$. □

Example B Let $X = [\alpha, \beta] \subset \mathbb{R}$ with the metric $d(x, y) = |x - y|$, where $\beta > \alpha > 0$. Let $F\colon X \times X \to \mathbb{R}$ be defined by $F(x, y) = x^2 - y^2$, then it is easy to see that for each $y \in X$, the function $x \to F(x, y)$ is a l.s.c. and convex function on X and $\inf_{x\in X} \sup_{y\in X} F(x, y) = \sup_{y\in X} \inf_{x\in X} F(x, y) = 0$. Note that for each $x \in (\alpha, \beta]$, $F(x, y) = x^2 - y^2 > \alpha^2 - y^2 = \inf_{a\in X} F(a, y)$ for all $y \in X$. Hence $X = \{u \in X : F(x, u) > \inf_{a\in X} F(a, u)\} \neq \emptyset$ for all $x \in (\alpha, \beta]$. For any $x, y \in (\alpha, \beta]$ with $x > y$, we have

$$d(x, y) = x - y < \frac{1}{2\alpha}(x^2 - y^2) = \frac{1}{2\alpha}(F(x, u) - F(y, u))$$

for all $u \in X$. Define a ℓ-function $p\colon X \times X \to \mathbb{R}$ by $p(x, y) = 2\alpha d(x, y)$. Thus $p(x, y) < F(x, u) - F(y, u)$ for all $x, y \in (\alpha, \beta]$ with $x > y$ and all $u \in X$. By Theorem 6.4, we also show that $\inf_{x\in X} \sup_{y\in X} F(x, y) = \sup_{y\in X} \inf_{x\in X} F(x, y)$.

Applying Theorem 4.4, we establish a generalization of Schauder's fixed point theorem.

Theorem 6.5 *Let $(X, \|\cdot\|)$ be a normed vector space and $T: X \to X$ be a continuous map. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \setminus K$ there exists $z \in M$ such that $\|z - Ty\| < \|y - Ty\|$. Then T has a fixed point in X.*

Proof Define $p: X \times X \to (-\infty, \infty]$ by

$$p(x, y) = \|y - Tx\| - \|x - Tx\|.$$

Then it is easy to see that p is a ℓ-function. By Theorem 4.4, there exists $v \in X$ such that $p(v, x) \geq 0$ or $\|x - Tv\| \geq \|v - Tv\|$ for all $x \in X$. Since $Tv \in X$, we have

$$0 \leq \|v - Tv\| \leq \|Tv - Tv\| = 0.$$

This implies that $Tv = v$ and v is a fixed point of T. □

Corollary 6.1 (Schauder's fixed point theorem) *Let X be a compact convex subset of a normed vector space $(E, \|\cdot\|)$ and $T: X \to X$ be a continuous map. Then T has a fixed point in X.*

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